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OVERSHOOTING IN STARS: FIVE OLD FALLACIES AND A NEW MODEL

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ABSTRACT

First, I discuss five fallacies of previous theoretical attempts to quantify overshooting in stars. I then suggest that the only reliable procedure is to solve the five dynamic equations for convective flux, kinetic and potential energy, turbulent pressure and dissipation rate of kinetic energy, plus the mean temperature equation. Finally, I discuss a recent solution of the new model that agrees with the latest helioseismological OV data and suggest that the same data be used to check the validity of the Schwarzschild criterion.
Subject headings: convection — stars: kinematics — turbulence

1. THE PROBLEM

Overshooting (OV) is a physical process that occurs in stars when heat is transported by convection. Unlike laboratory experiments, in stars there are no rigid walls, and when buoyancy forces vanish, the eddy acceleration vanishes but not the velocity which allows the eddies to "overshoot" into the adjacent radiative regions. In massive stars with a large convective core, the OV phenomenon acquires great relevance because it brings material with bigger mean molecular weight μ from the core into regions of lower μ. This affects the luminosity L which depends sensitively on μ, $L \sim \mu^4$–$\mu^{7.5}$. In turn, this affects the position of the star in the H-R diagram and, ultimately, age determinations. The OV problem can be stated as follows: *theoretical determinations of the OV extent are 5–20 times larger than what is required by observations.* Specifically, early and recent determinations based on stellar structure calibrations yield an OV extent that is a small fraction of the pressure scale height H_p; specifically, OV $\le \frac{1}{5} H_p$ (Prather & Demarque 1974; Andersen, Nordstrom, & Clausen 1990; Stothers & Chin 1991; Schaller et al. 1992; Nordstrom, Andersen, & Andersen 1997; Kozhurina-Platais et al. 1997). In the solar case, Basu, Antia, & Narashima (1994) obtained OV $= \frac{1}{10} H_p$, while Roxburgh & Vorontsov (1994) derived OV $= \frac{1}{4} H_p$. The most recent analysis yields OV $= \frac{1}{20} H_p$ (Basu 1997). On the other hand, theoretical determinations based on both modeling of turbulent convection (e.g., Xiong, Cheng, & Deng 1997; Roxburgh 1978) and direct numerical simulations (e.g., Singh, Roxburgh, & Chan 1995) yield large values for the OV, specifically, OV $\sim H_p$. Thus, there is a large discrepancy between theory and observations. In this Letter, we discuss three issues: first, we analyze five fallacies of past modeling of convection, second, we suggest a new model, and, third, we discuss a specific result for the Sun. We must stress that turbulence modeling is the only usable procedure since the alternative, numerical simulations, because of their computational requirements, cannot be linked to a stellar structure/evolution code.

2. THE BASIC EQUATIONS

Determining the extent of the OV (Fig. 1) is equivalent to determining the behavior of the convective flux F_c in the z-direction. (Since the extent of the OV is small compared to the overall convective zone, we use a one-dimensional model.) In the OV region, F_c is negative and vanishes at point C. The distance CB is the OV extent. If we call w and θ the fluctuating velocity and temperature fields, the convective flux is defined as $F_c = c_p \rho \overline{w\theta}$, where c_p is the specific heat at constant pressure, ρ is the density, and the overbar means an ensemble average. Consider the function $J = \overline{w\theta}$. Since it entails w and θ, one may suspect that one must also know $\overline{w}^2$ and $\overline{\theta}^2$ (kinetic and potential energy) and, thus, one needs three dynamic equations for J, $\overline{w}^2$, and $\overline{\theta}^2$. In fact, if we consider the first terms in the velocity and temperature equations

$$\frac{\partial w}{\partial t} = g\alpha\theta - \frac{\partial p}{\partial z} + \ \ldots, \tag{1a}$$

$$\frac{\partial \theta}{\partial t} = -w\beta + \chi \frac{\partial^2 \theta}{\partial z^2} + \ \ldots, \tag{1b}$$

where $\beta = TH_p^{-1}(\nabla - \nabla_{\rm ad})$ is the superadiabatic gradient, χ is the radiative conductivity, and g is the local gravity, we see that to construct the equation for J, one multiplies equation (1a) by θ, equation (1b) by w, averages and sums the two expressions. This brings in $\overline{w}^2$ and $\overline{\theta}^2$ for which one must construct the corresponding dynamic equations from equations (1a)–(1b). Actually, one needs more than three equations. The pressure gradient (the fluctuating pressure p does not obey the hydrostatic equilibrium equation) gives rise to the terms $\overline{w\partial p/\partial z}$ and $\overline{\theta\partial p/\partial z}$ which must be expressed in terms of the second-order moments (closure). In fact, they may be viewed as third-order moments since $p \sim \overline{w}^2$ is already a second-order moment. Velocity third-order moments exchange energy among eddies of different sizes, whereas pressure forces tend to isotropize the components u^2, v^2, and w^2 of an eddy of a given size (Batchelor 1971). As a dynamical process, $\overline{w\partial p/\partial z}$ occurs on the dynamical time scale τ_{pv}, while $\overline{\theta\partial p/\partial z}$ occurs on a time scale $\tau_{p\theta}$. The other time scales are τ and τ_θ, governing the rate of dissipation of kinetic and potential energy. For the derivation of the dynamic equations see Canuto (1992) and Canuto &

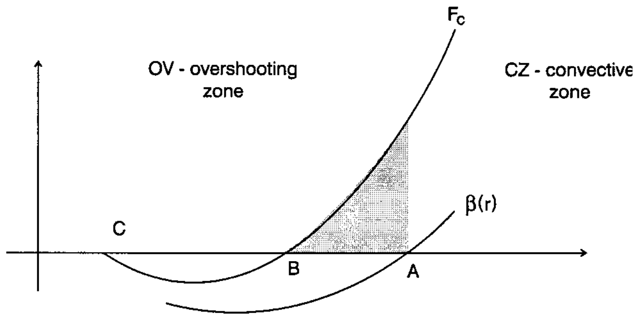

FIG. 1.—Sketch of the convective flux F_c and superadiabatic temperature gradient β below a convective zone (CZ): $\beta(r)$ goes to zero earlier than $F_c(r)$, which remains positive in the region AB as the result of a surplus of potential energy. In the AB region, the flux F_c is positive in spite of $\nabla - \nabla_{\rm ad} < 0$, in contradiction to the Schwarzschild criterion.

Dubovikov (1996, 1997a, 1998). The results are

$$\frac{\partial K}{\partial t} + D_f(K) = g\alpha J - \epsilon, \tag{2a}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{\theta}^2\right) + D_f\left(\frac{1}{2}\overline{\theta}^2\right) = \beta J - \tau_\theta^{-1}\overline{\theta}^2 + \frac{1}{2}\chi\frac{\partial^2}{\partial z^2}\overline{\theta}^2, \tag{2b}$$

$$\frac{\partial}{\partial t}J + D_f(J) = \beta\overline{w}^2 + \frac{2}{3}g\alpha\overline{\theta}^2 - \tau_{p\theta}^{-1}J + \frac{1}{2}\chi\frac{\partial^2 J}{\partial z^2}, \tag{2c}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{w}^2\right) + D_f\left(\frac{1}{2}\overline{w}^2\right) = -\tau_{pv}^{-1}\left(\overline{w}^2 - \frac{2}{3}K\right) + \frac{1}{3}(1+2\beta_5)g\alpha J - \frac{1}{3}\epsilon, \tag{2d}$$

$$\frac{\partial \epsilon}{\partial t} + D_f(\epsilon) = c_1\frac{\epsilon}{K}g\alpha J - c_2\frac{\epsilon^2}{K}. \tag{2e}$$

Here $D_f(\Lambda)$ represents the diffusion of Λ, e.g., $D_f(K) = \partial F_{\kappa e}/\partial z$, where $F_{\kappa e} = K\overline{w}$ is the flux of turbulent kinetic energy K; c_1, c_2, and β_5 are numerical constants, and $\alpha \equiv T^{-1}$ for a perfect gas. The relation of $\tau_{p\theta}$, τ_{pv}, and τ_θ to $\tau \equiv 2K/\epsilon$ is fully discussed in the above references. In addition, one has the equation for the mean temperature (F_r is the radiative flux)

$$c_p\frac{\partial T}{\partial t} + \frac{\partial K}{\partial t} = -\frac{\partial}{\partial z}\left[\frac{1}{\rho}(F_r + F_c + F_{\kappa e})\right]. \tag{2f}$$

This gives a total of six differential equations. The solution yields $J(r)$ and, thus, the extent of the OV region where $J < 0$. Given these general arguments, it is hard to conceive that one could reduce equations (2a)–(2f) to a single "criterion," the solution of which yields the OV.

3. FIRST FALLACY: "ONE-EQUATION" OV CRITERION

The prototype "one-equation" model is the expression (Roxburgh 1978)

$$\int_{r_1}^{r_2} T^{-2}\left|\left(\frac{\partial T}{\partial r}\right)_{\rm ad}\right|(L_N - L_r)dr = 0. \tag{3a}$$

Here L_N and L_r are the nuclear and radiative luminosities, and r_1 and r_2 are the beginning and end points of the CZ + OV regions. Since in the OV region $L_N - L_r < 0$, while in the CZ proper $L_N - L_r > 0$, the integrand changes sign at, say, r_* and the distance $r_2 - r_*$ is the OV extent. *Can one use equation (3a) or, alternatively, under what conditions does equation (3a) hold true?* Consider equations (2a) and (2f). If one eliminates J between the stationary versions of equations (2a) and (2f), the result is a differential equation for F_{ke} which can be integrated analytically. Introducing spherical coordinates and the luminosities $4\pi r^2 F = L$, one derives

$$\int_{r_1}^{r_2} T^{-2}\left|\left(\frac{\partial T}{\partial r}\right)_{\rm ad}\right|(L_N - L_r)\Phi(r)dr = \int_{r_1}^{r_2} 4\pi r^2 T^{-1}\rho\epsilon(r)\Phi(r)dr, \tag{3b}$$

where

$$\ln\Phi(r) = -\int T^{-1}\beta(r)dr. \tag{3c}$$

Relation (3b) is exact but useless unless we know $\epsilon(r)$ and $\beta(r)$. To know $\epsilon(r)$, we must solve equation (2e), which entails K, which requires that we solve equation (2a), which entails J, etc. In deriving equations (3b) and (3c) we have done algebra, not physics. We may have gained an insight into what it takes to quantify the OV, but we have not gained any operational advantage over equations (2a)–(2f). However, if one assumes that

$$\epsilon(r) = 0, \quad \beta(r) = 0, \tag{3d}$$

equation (3b) reduces to equation (3a), which is operationally deterministic. The question then arises: is equation (3d) correct? Even though sometimes approximations that are individually incorrect cancel each other, rendering them more palatable, at least at the pragmatic level, this is not the case here: *both assumptions in equation (3d) lead one to overestimate the OV*; their effects do not cancel, instead they add up. To see this, we divide the interval $r_2 - r_1$ into $r_* - r_1$ and $r_2 - r_*$, where r_* has been defined above. After some steps we have

$$\int_{r_1}^{r^*} T^{-2}\frac{g}{c_p}\,|\,L_N - L_r\,|\,\Phi_1(r)dr = \int_{r_*}^{r_2} T^{-2}\frac{g}{c_p}\,|\,L_N - L_r\,|\,\Phi_2(r)dr + D, \tag{4a}$$

where D represents the dissipation term in the right-hand side of equation (3b). Moreover,

$$\Phi_1 = \exp\left(-\int T^{-1} \mid \beta(r) \mid dr\right) \approx 1,$$
$$\Phi_2 = \exp\left(\int T^{-1} \mid \beta(r) \mid dr\right) > 1. \tag{4b}$$

If we take $\epsilon(r) = 0$, the right-hand side of equation (4a) will be deprived of the positive contribution D. Thus, to balance the left-hand side, one is forced to take a large upper limit r_2, overestimating the OV. Next, consider $\beta(r) = 0$. Since $\Phi_2 > 1$, the integrand in the right-hand side is larger than it would be with $\Phi_2 = 1$, while the integrand on the left-hand side is smaller than it would be with $\Phi_1 = 1$. Thus, $\Phi_2 \neq 1$ does not require an r_2 as large as in the $\Phi_2 = 1$ case, which, once again, leads to overestimate the OV. In conclusion, both assumptions in equation (3d) lead one to overestimate the OV, and equation (3a) can at best give the upper limit to the OV extent.

4. SECOND FALLACY: $F_c = 0$ CORRESPONDS TO $\nabla - \nabla_d = 0$ (THE SCHWARZSCHILD CRITERION)

Consider Figure 1 where at A, $\beta(r)$ vanishes and the CZ is to the right of A. The region to the left of B, where F_c is negative, is the OV region proper. The region AB is characterized by $\beta(r) < 0$ and yet $F_c > 0$. The physics is clear if we take the stationary limit of equation (2c) and neglect χ for the moment. Simple algebra gives (P = potential energy),

$$J = \chi_t \beta, \tag{5a}$$

$$\chi_t \sim \tau_{p\theta} K[1 + \mathrm{sgn}(\beta)P/K], \quad P \equiv \frac{1}{2} g\alpha \mid \beta \mid^{-1} \overline{\theta}^2, \tag{5b}$$

where χ_t is a turbulent diffusivity. In the AB region, $\beta < 0$, and the sign of χ_t depends on the ratio P/K. Even when $\beta < 0$, F_c can still be positive if $P > K$. This is indeed what happens in the AB region: *the $\overline{\theta}^2$ fluctuations are large, and they act as a countergradient and contribute to the convective flux more than the kinetic energy*. This means that the convective flux does not vanish when $\nabla - \nabla_{\mathrm{ad}} = 0$, as required by the Schwarzschild criterion. It would be interesting to know if helioseismic data exhibit the region AB (which could be as much as ~20% of the OV region) which does not satisfy the Schwarzschild criterion (Canuto & Christensen-Dalsgaard 1998).

5. THIRD FALLACY: $\beta(r) = 0$ IN THE OV REGION

Since in the region around A convection is still efficient, equations (2a) and (2b) may be taken as

$$\frac{\partial K}{\partial t} = g\alpha J - \epsilon, \tag{5c}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{\theta}^2\right) = (\beta(r)J - \tau_\theta^{-1}\overline{\theta}^2. \tag{5d}$$

Suppose that at $t = 0$, we begin with $K(t = 0) \neq 0$ but no potential energy, $\overline{\theta}^2(t = 0) = 0$. Since the latter must grow, equation (5d) tells us that the necessary condition is $\beta(r) \neq 0$. Conversely, if at $t = 0$ we begin with potential energy (density fluctuations since $\theta \sim \delta\rho/\rho$) rather than kinetic energy, the latter must increase, which implies $J > 0$. Thus, no matter whether we begin with velocity or density fluctuations, J fluctuates in time between positive and negative values with the Brunt-Vaisala frequency $N = [ga \mid \beta(r) \mid]^{1/2}$. This phenomenon has been confirmed by detailed numerical simulations (Gerz, Schumann, & Elghobashi 1989), and it occurs only if $\beta(r) \neq 0$. Stationarity is achieved when the diffusion terms, acting like a source, become important to the left of B.

6. FOURTH FALLACY: $\epsilon(r) = 0$ IN THE OV REGION (ENERGY CONSERVATION)

Since the dissipation of a velocity field can occur only via kinematic viscosity, ϵ represents the rate of dissipation of K by viscous forces and is defined as

$$\epsilon = 2\nu \left(\frac{\partial u_i}{\partial x_i}\right)^2 = 2\nu\Omega = 2\nu \int k^2 E(k) dk, \tag{6}$$

where Ω is the enstrophy $2\Omega = \overline{\omega}^2$ (ω is the vorticity) and $E(k)$ is the eddy energy spectrum. Since in stellar interiors ν is 10 orders of magnitude smaller than the radiative conductivity χ, it is often implied that one can neglect ϵ since ν is small. This is not so. The argument is as follows: the nonlinear interactions conserve energy, which means that the energy at the largest scales must be dissipated in its entirety by viscous forces. *Energy conservation requires that ϵ cannot be zero*. This argument is independent of viscosity, and yet equation (6) seems to imply that ϵ depends on ν. The correct interpretation is that ϵ is indeed independent of ν, and what equation (6) tells us is at which wavenumber k (or scale), the dissipation process occurs. It should not be viewed as giving us the amount of energy that must be dissipated (because we do not know the spectrum in the high k region) since ϵ is the same as the energy input. The smaller the ν, the smaller the scales at which dissipation occurs: as $\nu \to 0$, $\Omega \to \infty$, *and the product is constant*. That ϵ is independent of ν is well known in turbulence. The argument that $\nu \to 0$ implies $\epsilon \to 0$ is mathematically and physically incorrect.

7. FIFTH FALLACY: LOCAL EXPRESSION FOR ϵ

We are not aware of any OV calculations which adopt equation (2e). The most widely used expression for ϵ is a local one (Gough 1976; Xiong, Cheng, & Deng 1997; Balmforth 1992). It is obtained from equation (2e) by taking

$$D_f(\epsilon) = -\frac{\partial}{\partial z} w\epsilon \to l^{-1}\overline{w\epsilon} \to l^{-1}K^{1/2}\epsilon, \tag{7a}$$

where l is a mixing length. Once used in equation (2e), it gives the well-known local expression

$$\epsilon = l^{-1}K^{3/2}. \tag{7b}$$

Use of equation (7b) leads to divergent results, as discussed recently (Canuto & Dubovikov 1997b).

8. CONCLUSIONS

The hope to quantify the OV with a single expression, a criterion so to speak, has remained regrettably unfulfilled. Equation (3b) is nothing more than a mathematical rewriting of two of the six differential equations (2a)–(2f). It contains no new physics and, thus, no more information than the original equations. To use equation (3b), one must know two ingredients, $\epsilon(r)$ and $\beta(r)$, which require the solutions of the remaining equations; thus, equation (3b) offers no operational or practical advantages. One can artificially disentangle equation (3b) from the rest of the equations by taking $\epsilon = 0$ and $\beta = 0$. These assumptions are flawed since $\epsilon = 0$ is in violation of energy conservation. At the more general level, since equation (3b) with $\epsilon = 0$ and $\beta = 0$ is a relation in which turbulence has disappeared, one may wonder how can it be used to infer a turbulence property like the OV? There seem to be no alternatives or shortcuts to solving equations (2a)–(2f). New results (Antia & Basu 1997, private communication) indicate that in the case of the Sun, equations (2a)–(2f) predict an OV smaller than the upper limit of $\frac{1}{20}\,H_p$ set by helioseismology. The hope is that the same model will now be applied to the case of massive stars.

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